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THEORY OF THE DIPOLE ANTENNA AND THE TWO-WIRE TRANSMISSION LINE

By

Tai Tsun Wu

March 10, 1960

Technical Report No. 318

Cruft Laboratory
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TR 313

Theory of the Dipole Antenna and the Two-Wire Transmission Line *

by

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Cambridge, Massachusetts

Abstract

The properties of the dipole antenna are studied by an approximate procedure that makes use of the Wiener-Hopf integral equation. In particular, the input admittance and the radiation pattern are found. Simple formulas are obtained only when the dipole antenna is more than one wavelength long. The present results thus supplement the existing theories, which are concerned mostly with shorter dipoles.

The same procedure is then applied to several related problems. First, the back-scattering cross section of a dipole antenna is found approximately for normal incidence. Secondly, the two-wire transmission line is studied in detail by considering it to be two coupled dipole antennas. The capacitive end-correction for an open end is evaluated, and the radiated power and the radiation resistance are found for a resonant section of transmission line with both ends open. Finally, the dielectric-coated antenna is considered briefly.

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Introduction

The problem of the center-driven cylindrical antenna has been investigated by numerous authors. There exist now principally three kinds of attacks: iterative procedures, variational methods, and Fourier series expansions. Recently, Duncan and Hinchey [1] used the last method to get some extremely interesting results. To employ this method, it is essential to carry out the calculations on a high-speed digital computer. The other two kinds of methods are described in detail in the monumental book of King [2], which will be designated by K in subsequent references. For convenience, numerous references will be made to this book instead of the original papers.

For thin dipole antennas of length not much more than one wavelength, the King-Middleton iterative procedure [K, p. 101 ff] yields current distributions in good agreement with the experimental results. So far as the input impedance is concerned, the various iterative and variational methods seem to give comparable results for thin antennas not much more than two wavelengths in total length [K, p. 843]. If $\underline{h}$ is the half-length of the thin antenna, it is reasonable to think that cases where $\underline{h} \lesssim \lambda$ are fairly well understood.

The situation is much less favorable for $\underline{h} > \lambda$. Both theoretical and experimental results are very scarce in this vast range of antenna lengths. So far as the author is aware, the following three pieces of information are available:

1. First-order King-Middleton distributions of currents for antennas with $\underline{h} = 6\lambda$ [K, p. 115],
2. Robert's measurement of current distribution for an antenna with $\underline{h} \sim 11\lambda$ [K, p. 140], and

3. Altshuler's [3] recent measurements of current distributions for antennas with $\underline{h} \sim 2\lambda$.

No general conclusions can be drawn from these results, except that there is essentially no evidence of any agreement between theory and experiment. A more direct comparison is possible in the related problem of the determination of the back-scattering cross sections of a dipole receiving antenna at normal incidence [K, p. 508, p. 516]. Here the theoretical and experimental results are certainly in disagreement for $\underline{h} \gtrsim 0.8\lambda$.

If these few pieces of experimental data are not dismissed as incorrectly recorded, then one is forced to consider the possibility that the existing theories of the dipole antenna may be inapplicable when $\underline{h} \gtrsim \lambda$. An iterative procedure is one which gives correction terms to an initial, rough approximation of the current distribution. Since usually only a small number of iterations can be carried out, the accuracy of the results depends critically on the accuracy of the initial approximation. Since the iterative procedure itself can hardly be questioned, an explanation of the discrepancy between theory and experiment may be sought in the inadequacy of the initial approximation.

This paper is devoted to the following question: Starting from the integral-equation formulation of the antenna problem, how can one get a rough approximation to the various characteristics of the antenna? This is properly the first step of an iterative procedure. However, unless this approximation is of an unrealistically simple form, no iteration can be conveniently carried out. Therefore, in determining this rough approximation, it should be kept in mind that the result must be at least semi-quantitatively

correct before it is of any interest or of any use. A possible answer to this question is given in Part I for the center-driven dipole antenna. In Part II, the same procedure is applied to a few somewhat more complicated cases. The interest is mostly centered around the situation when the dipole antenna is several wavelengths long.

Throughout this paper, only symmetrical, center-driven antennas are treated. The generalization to asymmetrical cases seems to offer no difficulty in principle.

Part I. The Center-Driven Dipole Antenna

1. Formulation of the Problem

The dipole antenna is assumed to be a symmetric tubular antenna of zero thickness and infinite conductivity defined by $r = a$, $|z| \leq h$, as shown in Fig. 1. If $I(z)$ is the total z-component of the current at z, including both the current on the outside of the tube and that on the inside, then

$$I(h) = 0, \tag{1.1}$$

and the z-component of the vector potential on the cylinder $r = a$ is given by

$$A(z) = \frac{\mu_o}{4\pi} \int_{-h}^{h} dz'\, I(z')\, K(z - z'), \tag{1.2}$$

where μ_o is the free-space magnetic permeability and the kernel K is given by

$$K(z) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\theta\, [z^2 + (2a \sin\theta/2)^2]^{-\frac{1}{2}} \exp\{ik[z^2 + (2a \sin\theta/2)^2]^{\frac{1}{2}}\}. \tag{1.3}$$

Here k is the wave number. As usual in antenna theory, the term "vector potential" is used to denote the vector potential in the Lorentz gauge satisfying the Sommerfeld radiation condition. On the other hand, if the strength of the δ-function generator is taken to be -1, the z-component of the vector potential is of the form

$$A(z) = \frac{\mu_o i}{\zeta_o} [C \cos kz + \frac{1}{2} \sin k|z|] \tag{1.4}$$

for $|z| < h$. Here ζ_o is the characteristic impedance of free space. The

combination of (1.2) and (1.4) gives the integral equation for $\underline{I}(\underline{z})$, where $\underline{C}$ is to be determined by the boundary condition (1.1).

In the King-Middleton iterative solution, the vector potential is assumed to be proportional to the current at the same point in order to get the initial rough approximation. This assumption is reasonable except near the ends of the dipole antenna, where the current changes rapidly with $\underline{z}$. Unfortunately, the value of the constant $\underline{C}$ is determined at $\underline{z} = \underline{h}$, precisely where this approximation is poor. It is here proposed to find $\underline{C}$ by a different procedure, making use of the observation that $\underline{A}(\underline{z})$ is relatively small for $|\underline{z}| > \underline{h}$ [K, p. 429, p. 527]. This observation is useful because then the antenna may be approximated by a semi-infinite one driven by a vector potential distribution which is of the form (1.4) for $|\underline{z}| < \underline{h}$ and is zero for $\underline{z} > \underline{h}$. Consequently, the problem of the semi-infinite antenna is to be studied first.

2. The Semi-Infinite Antenna

The semi-infinite antenna is described by an integral equation of the Wiener-Hopf type:

$$\int_0^\infty d\,z'\, I(z')\, K(z - z') = F(z), \tag{2.1}$$

where $\underline{F}(\underline{z})$ is known for $\underline{z} > 0$. It is assumed that $\underline{F}(0+)$ and $\underline{F}'(0+)$ both exist, and $\underline{F}(\underline{z})$ approaches zero sufficiently rapidly as $\underline{z} \to \infty$. Under these circumstances, $\underline{I}(\underline{z})$ is in general unbounded near $\underline{z} = 0$. If, however, $\underline{F}(\underline{z})$ satisfies a certain integral condition, $\underline{I}(\underline{z})$ becomes bounded near $\underline{z} = 0$, and furthermore $\lim_{\underline{z} \to 0} \underline{I}(\underline{z}) = 0$. It is desired to find this integral condition.

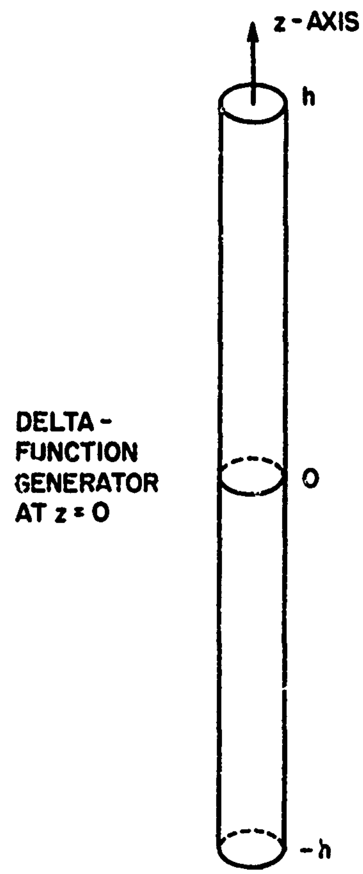

FIG. 1 THE CENTER-DRIVEN DIPOLE ANTENNA OF HALF-LENGH h

For this purpose, the usual Wiener-Hopf procedure may be used.

Consider $\underline{k}$ to have a small positive imaginary part which is eventually allowed to approach zero:

$$\mathrm{Im}\, k = \epsilon > 0 \,, \tag{2.2}$$

and define the relevant Fourier transforms by

$$\overline{I}(\zeta) = \int_0^{\infty} dz\, I(z) \exp(-i\zeta z) \,, \tag{2.3a}$$

$$\overline{K}(\zeta) = \int_{-\infty}^{\infty} dz\, K(z) \exp(-i\zeta z) = \pi i J_o\left[a(k^2 - \zeta^2)^{\frac{1}{2}}\right] H_o^{(1)}\left[a(k^2 - \zeta^2)^{\frac{1}{2}}\right] \,, \tag{2.3b}$$

$$\overline{F}_-(\zeta) = \int_0^{\infty} dz\, F(z) \exp(-i\zeta z) \,, \tag{2.3c}$$

and

$$\overline{F}_+(\zeta) = \int_{-\infty}^{0} dz\, F(z) \exp(-i\zeta z) \,. \tag{2.3d}$$

Then (2.1)' leads to

$$\overline{I}(\zeta)\,\overline{K}(\zeta) = \overline{F}_-(\zeta) + \overline{F}_+(\zeta) \,. \tag{2.4}$$

For any function $\underline{f}$ of ζ analytic in the strip $|\mathrm{Im}\,\zeta| < \epsilon$, define

$$[f(\zeta)]_- = -(2\pi i)^{-1} \int_{-\infty + i\epsilon/2}^{\infty + i\epsilon/2} d\zeta'\, (\zeta' - \zeta)^{-1} f(\zeta') \,, \tag{2.5a}$$

and

$$[f(\zeta)]_+ = (2\pi i)^{-1} \int_{-\infty - i\epsilon/2}^{\infty - i\epsilon/2} d\zeta'\, (\zeta' - \zeta)^{-1} f(\zeta') \,, \tag{2.5b}$$

where the Cauchy principal values are taken at ∞. Then $[f(\zeta)]_-$ is analytic for $\mathrm{Im}\,\zeta < \epsilon/2$, and $[f(\zeta)]_+$ is analytic for $\mathrm{Im}\,\zeta > -\epsilon/2$. Furthermore, in the strip $|\mathrm{Im}\,\zeta| < \epsilon/2$,

$$f(\zeta) = [f(\zeta)]_- + [f(\zeta)]_+ . \tag{2.6}$$

In terms of the function $\bar{K}(\zeta)$, define

$$\bar{L}_\pm(\zeta) = \exp[\mp \ln \bar{K}(\zeta)]_\pm , \tag{2.7}$$

then

$$\bar{K}(\zeta) = \bar{L}_-(\zeta)/\bar{L}_+(\zeta) \tag{2.8}$$

for $|\mathrm{Im}\,\zeta| < \epsilon/2$, and furthermore,

$$\bar{L}_-(\zeta) = [\bar{L}_+(-\zeta)]^{-1} . \tag{2.9}$$

From (2.4) and (2.8) it follows that

$$\bar{I}(\zeta)\bar{L}_-(\zeta) - [\bar{F}_-(\zeta)\bar{L}_+(\zeta)]_- = [\bar{F}_-(\zeta)\bar{L}_+(\zeta)]_+ + \bar{F}_+(\zeta)\bar{L}_+(\zeta). \tag{2.10}$$

This defines an entire function, which must be zero because of the behavior at infinity. Therefore,

$$\bar{I}(\zeta)\bar{L}_-(\zeta) = [\bar{F}_-(\zeta)\bar{L}_+(\zeta)]_- . \tag{2.11}$$

It is assumed that $\underline{I}(\underline{z})$ has no singularity at $\underline{z} = 0$. Thus, as $|\zeta| \to \infty$ in the half plane $|\mathrm{Im}\,\zeta| < \epsilon/2$,

$$\bar{I}(\zeta) = O(|\zeta|^{-1}), \tag{2.12}$$

and furthermore

$$\overline{L}_-(\zeta) = O(|\zeta|^{-\frac{1}{2}}) . \quad (2.13)$$

It, therefore, follows from (2.11) that

$$[\overline{F}_-(\zeta)\,\overline{L}_+(\zeta)]_- = 6(|\zeta|^{-1}), \quad (2.14)$$

as $|\zeta| \to \infty$. This is the required condition on $\underline{F}(\underline{z})$.

In order to put (2.14) in a simpler form, let $\epsilon \to 0+$ and note that $\overline{K}(\zeta)$ is analytic in the cut plane as shown in Fig. 2. Furthermore, the contours used in defining $[\underline{f}(\underline{z})]_{\pm}$ both become $\underline{C}_o$, also shown in Fig. 2, and $\overline{L}_+(\zeta)$ and $\overline{L}_-(\zeta)$ are analytic in the entire complex plane except respectively the left and the right branch cuts of $\overline{K}(\zeta)$. Define

$$\overline{F}_-^{\,o}(\zeta) = \overline{F}_-(\zeta) - \frac{F(0+)}{i(\zeta+k)} , \quad (2.15)$$

then $\overline{F}_-^{\,o}(\zeta)\,\overline{L}_+(\zeta)$ is integrable along $\underline{C}_o$, at least in the sense of Euler summability. Therefore, the left-hand side of (2.14) is explicitly

$$[\overline{F}_-(\zeta)\,\overline{L}_+(\zeta)]_- = F(0+)\,[\frac{1}{i(\zeta+k)}\,\overline{L}_+(\zeta)]_- + [\overline{F}_-^{\,o}(\zeta)\,\overline{L}_+(\zeta)]_-$$

$$= -(2\pi i)^{-1} \int_{C_o} d\zeta'\,(\zeta'-\zeta)^{-1}\,\overline{F}_-^{\,o}(\zeta)\,\overline{L}_+(\zeta) . \quad (2.16)$$

The condition (2.14) is thus explicitly

$$\int_{C_o} d\zeta \;\; \overline{F}_-^{\,o}(\zeta)\,\overline{L}_+(\zeta) = 0. \quad (2.17)$$

3. Approximations for Thin Antennas

The condition (2.17) for the vanishing of the current at the end of the semi-infinite antenna is exact. This rather complicated condition can be greatly simplified if it is assumed (1) that the antenna is thin in the sense that $a/\lambda \ll 1$ and (2) that the characteristic distance for the variation of $F(z)$ is much larger than a. Under the second assumption, which is satisfied if the right-hand side of (1.4) is taken to be $F(h-z)$, the behavior of $\overline{L}_+(\zeta)$ for $\zeta \sim a^{-1}$ is unimportant, and thus it is permissible to use the following approximation of $\overline{K}(\zeta)$:

$$\overline{K}(\zeta) = 2\Omega_1 - \ln\left[(k^2 - \zeta^2)/k^2\right] , \tag{3.1}$$

where

$$\Omega_1 = \Omega_o + i\pi/2 \tag{3.2}$$

and

$$\Omega_o = \ln(2/ka) - \gamma . \tag{3.3}$$

In (3.3), γ is Euler's constant, numerically about 0.57722. In view of a previous discussion[4] on the meaning of the input admittance of a linear antenna driven by a delta function, the approximation (3.1) does not introduce any further error beyond those inherent in the model of the delta-function generator. Once (3.1) is used, it is possible to define

$$\overline{M}(\zeta) = [\overline{K}(\zeta)]^{-1}, \tag{3.4}$$

$$\overline{M}(\zeta) = \int_{-\infty}^{\infty} dz\, M(z) \exp(-i\zeta z) , \tag{3.5}$$

and

$$\overline{L}_+(\zeta) = \int_{-\infty}^{\infty} dz\, L_+(z) \exp(-i\zeta z) . \tag{3.6}$$

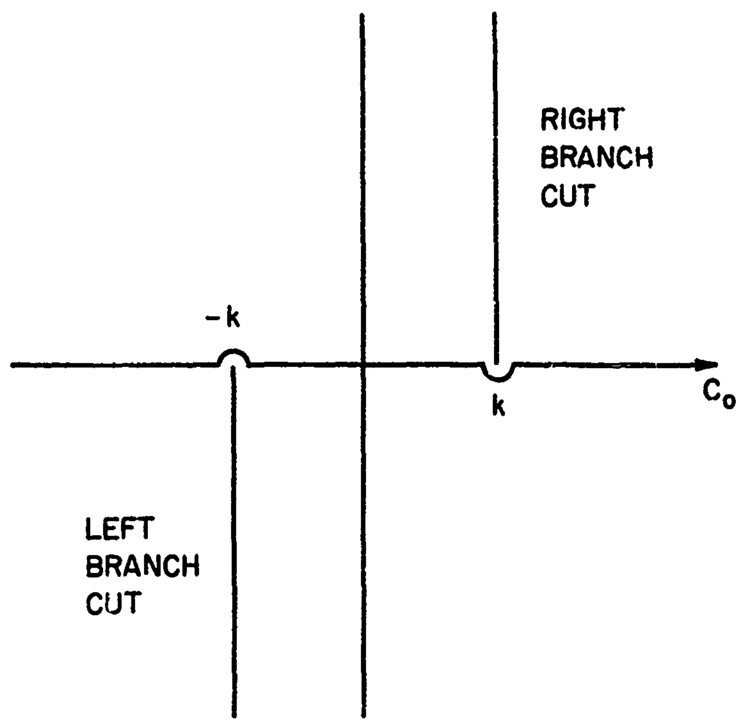

FIG. 2 THE ζ-PLANE AND THE CONTOUR C_o

It should be emphasized that $\underline{M}(z)$ and $\underline{L}_+(z)$ are meaningful only when an approximation of the type (3.1) is used. The exact $\bar{M}(\zeta)$ and $\bar{L}_+(\zeta)$ are not the Fourier transforms of integrable functions.

First, an approximation is to be found for $\underline{M}(\underline{z})$, which is given by

$$M(z) = ik(2\pi)^{-1}\, e^{ikz} \int_0^\infty d\xi\, e^{-kz\xi} \left(\left\{ 2\Omega_0 - \ln\left[\xi(2+i\xi)\right] - \frac{i\pi}{2} \right\}^{-1} - \left\{ 2\Omega_0 - \ln\left[\xi(2+i\xi)\right] + \frac{3i\pi}{2} \right\}^{-1} \right) \tag{3.7}$$

for $\underline{z} > 0$. When $\underline{z}$ is not too small, it will be a reasonably good approximation to replace $\ln(2+\xi)$ in the integrand by $\ln 2$, and replace $\ln\xi$ by the average

$$\left[\int_0^\infty d\xi\, e^{-kz\xi} \right]^{-1} \int_0^\infty d\xi\, e^{-kz\xi} \ln\xi = -\ln(kz) - \gamma . \tag{3.8}$$

With these replacements, (3.7) becomes

$$M(z) = i(2\pi z)^{-1} e^{ikz} \left[\left(2\Omega_0 - \ln\frac{2}{kz} + \gamma - \frac{i\pi}{2}\right)^{-1} - \left(2\Omega_0 - \ln\frac{2}{kz} + \gamma + \frac{3i\pi}{2}\right)^{-1} \right]. \tag{3.9}$$

Therefore $\int_1^\infty |\underline{M}(\underline{z})|\underline{dz}$ exists. The same statement is true of $\underline{L}_+(-\underline{z})$.

If the integral equation had been written in terms of the electric field instead of the vector potential, the corresponding $\underline{M}$ and $\underline{L}_+$ functions would not have this property, i.e., they would not approach zero sufficiently rapidly. The rapid decrease of $\underline{M}$ and $\underline{L}_+$ implies that the behavior of $\underline{A}$ for $\underline{z} > \underline{h}$ is relatively unimportant in the present calculation. As will be seen in Sec. 8,

the condition that $\underline{M}$ and $\underline{L}_+$ should decrease sufficiently rapidly dictates the choice of the differential operator that gives $\underline{E}$ from $\underline{A}$.

4. Application to the Dipole Antenna

To apply the results of the last two sections to the dipole antenna, it is convenient to define for $\underline{Z} > 0$.

$$S(Z) = \int_{C_0} d\zeta\, \overline{M}(\zeta) \int_0^\infty dz \exp(-i\zeta z) \sin kz\, H(Z-z)\,, \tag{4.1}$$

$$T(Z) = \int_{C_0} d\zeta\, \overline{M}(\zeta) \int_0^\infty dz \exp(-i\zeta z) [\cos kz\, H(Z-z) - \exp(-ikz)]\,, \tag{4.2}$$

$$S'(Z) = \int_{C_0} d\zeta\, \overline{L}_+(\zeta) \int_0^\infty dz \exp(-i\zeta z) \sin kz\, H(Z-z)\,, \tag{4.3}$$

and

$$T'(Z) = \int_{C_0} d\zeta\, \overline{L}_+(\zeta) \int_0^\infty dz \exp(-i\zeta z) [\cos kz\, H(Z-z) - \exp(-ikz)]\,, \tag{4.4}$$

where H is the Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

By comparing (1.2) and (1.4) with (2.1), it is seen that for the dipole antenna and for $\underline{z} > 0$

$$F(z) = 4\pi i \zeta_0^{-1} [C \cos k(h-z) + \tfrac{1}{2} \sin k\,|h-z|]\, H(2h-z)\,, \tag{4.5}$$

which may be alternatively expressed as

$$F(z) = 4\pi i \zeta_o^{-1} [\sin kh \cos kz\, H(h - z) - \cos kh \sin kz\, H(h - z)$$

$$+ (C \cos kh - \frac{1}{2} \sin kh) \cos kz\, H(2h - z) + (C \sin kh + \frac{1}{2} \cos kh) \sin kh\, H(2h - z)]. \tag{4.6}$$

When this is substituted into (2.17), the result is, using (4.3, 4),

$$\sin kh\, T'(h) - \cos kh\, S'(h) + (C \cos kh - \frac{1}{2} \sin kh)\, T'(2h)$$

$$+ (C \sin kh + \frac{1}{2} \cos kh)\, S'(2h) = 0. \tag{4.7}$$

The constant C is thus explicitly given by

$$C = -\frac{1}{2} [\cos kh\, T'(2h) + \sin kh\, S'(2h)]^{-1} \Big\{ \sin kh\, [2T'(h) - T'(2h)]$$

$$- \cos kh\, [2S'(h) - S'(2h)] \Big\}. \tag{4.8}$$

On the other hand, it follows from (4.5) that the input admittance of the antenna is

$$Y = 2 i \zeta_o^{-1} [S(h) + C U(h)], \tag{4.9}$$

where

$$U(Z) = \int_{C_o} d\zeta\, \overline{M}(\zeta) \int_{-Z}^{Z} dz \exp(-i\zeta z) \cos kz. \tag{4.10}$$

In order to put (4.8) and (4.10) in forms that are easily computed, many approximations have to be made. In (4.1-4), the contour C_o can be deformed so that it is wrapped around the left branch cut. When kZ is not too small, the contributions to the four functions, insofar as the ζ-integral is

concerned, come mainly from the region $|\zeta + k| Z \lesssim 1$. On the other hand, from (2.8). $\mathcal{L}_+(\zeta)$ is the same as $M(\zeta)\,\mathcal{L}_-(\zeta)$, where $\mathcal{L}_-(\zeta)$ is analytic in the vicinity of $\zeta = -k$. Therefore, approximately

$$S'(Z) = \mathcal{L}_-(-k)\; S(Z)\,, \tag{4.11}$$

and

$$T'(Z) = \overline{\mathcal{L}}_-(-k)\; T(Z)\,. \tag{4.12}$$

Since dipole antennas with $kh \lesssim \pi$ are well understood in terms of the King-Middleton theory, the task here is thus to calculate $S(Z)$, $T(Z)$ and $U(Z)$ for kZ large. This is carried out in Appendix A, with the results

$$\begin{aligned} 2S(Z) = &- \ln\left[1 + \pi i\,(\Omega_o - \ln 2)^{-1}\right] - \frac{\pi^2}{12}\left[(\Omega_o - 2\ln 2)^{-2} - (\Omega_o - 2\ln 2 + \pi i)^{-2}\right] \\ &+ \ln\left\{[\Omega_2(Z)]^{-1}\,\Omega_3(Z)\right\} + \frac{1}{2}\gamma'\left\{[\Omega_2(Z)]^{-2} - [\Omega_3(Z)]^{-2}\right\} \\ &- i\,(2kZ)^{-1}\exp(2ikZ)\left\{[\Omega_2(Z)]^{-1} - [\Omega_3(Z)]^{-1}\right\}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} -2i\,T(Z) = &- \ln\left[1 + \pi i\,(\Omega_o - \ln 2)^{-1}\right] - \frac{\pi^2}{12}\left[(\Omega_o - 2\ln 2)^{-2} - (\Omega_o - 2\ln 2 + \pi i)^{-2}\right] \\ &- \ln\left\{[\Omega_2(Z)]^{-1}\,\Omega_3(Z)\right\} - \frac{1}{2}\gamma'\left\{[\Omega_2(Z)]^{-2} - [\Omega_3(Z)]^{-2}\right\} \\ &- i\,(2kZ)^{-1}\exp(2ikZ)\left\{[\Omega_2(Z)]^{-1} - [\Omega_3(Z)]^{-1}\right\}, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} i\,U(Z) = &\ln\left\{[\Omega_2(Z)]^{-1}\,\Omega_3(Z)\right\} + \frac{1}{2}\gamma'\left\{[\Omega_2(Z)]^{-2} - [\Omega_3(Z)]^{-2}\right\} \\ &+ i\,(2kZ)^{-1}\exp(2ikZ)\left\{[\Omega_2(Z)]^{-1} - [\Omega_3(Z)]^{-1}\right\}. \end{aligned} \tag{4.15}$$

In (4.13 - 15), the following symbols have been used:

$$\Omega_2(Z) = 2(\Omega_o - \ln 2) + \ln(2kZ) + \gamma - i\pi/2 , \qquad (4.16)$$

$$\Omega_3(Z) = 2(\Omega_o - \ln 2) + \ln(2kZ) + \gamma + 3i\pi/2 , \qquad (4.17)$$

and

$$\gamma' = \Gamma''(1) - \gamma^2 . \qquad (4.18)$$

γ' is numerically about 1.6449.

5. The Radiation Field

The procedure of Sec. 4 may be used to get the current distribution not close to either the generator or the ends. However, this point is not studied further because excessive numerical computation seems to be necessary.

Let a spherical coordinate system (r, θ, ϕ) be set up such that the ends of the antenna are at $(h, 0, \phi)$ and (h, π, ϕ). All field quantities are independent of ϕ because of rotational symmetry. Define the field pattern by

$$F(\theta) = -\lim_{r\to\infty} E_\theta(r, \theta)\, r \exp(-ikr) . \qquad (5.1)$$

By (1,2), this is

$$F(\theta) = i\omega\mu_o (4\pi)^{-1} \sin\theta \int_{-h}^{h} dz\, I(z)\, (2\pi)^{-1}$$

$$\int_{-\pi}^{\pi} d\theta' \exp[-ik(z\cos\theta - a\cos\theta' \sin\theta)] . \qquad (5.2)$$

When the small term proportional to <u>a</u> is neglected, this simplifies to

$$F(\theta) = i\omega\mu_o (4\pi)^{-1}\, \overline{I}\,(k\cos\theta)\sin\theta \quad . \tag{5.3}$$

Within the framework of the present approximation, this is simply

$$F(\theta) = -\frac{1}{2}\sin\theta[\Omega_1 - \ell n \sin\theta]^{-1} \left\{ C\,\frac{\sin[kh(1-\cos\theta)]}{1-\cos\theta} + C\,\frac{\sin[kh(1+\cos\theta)]}{1+\cos\theta} + \frac{1}{2}\,\frac{1-\cos[kh(1-\cos\theta)]}{1-\cos\theta} + \frac{1}{2}\,\frac{1-\cos[kh(1+\cos\theta)]}{1+\cos\theta} \right\} . \tag{5.4}$$

This differs from the usual zeroth-order field pattern only in the appearance of the factor $[\Omega_1 - \ell n \sin\theta]^{-1}$ when C is chosen in a sufficiently simple way. For a long dipole, this factor has the effect of reducing the end-firing major lobes.

Part II. Generalizations

6. Back-Scattering Cross Section

In this and the two following sections, the procedure of Part I is to be applied to three situations mathematically similar to the one already treated. The first problem is the determination of the back-scattering cross section of an unloaded dipole antenna at normal incidence. The geometry is shown in Fig. 3. Without loss of generality, the incident electric field is taken to be 1 at $x = 0$. Since, the radius $\underline{a}$ of the dipole antenna is assumed to be very small compared with the wavelength, the scattered field is considered to be rotationally symmetrical. Under this approximation, the current induced on the antenna satisfies the integral equation:

$$\int_{-h}^{h} dz' \, I_s(z') \, K(z-z') = 4\pi i \, (\mu_o \omega)^{-1} \, [1 + C_s \cos kz] \quad . \tag{6.1}$$

Here the subscript s is used to distinguish the present scattering problem, and the constant C_s is to be determined from the boundary condition.

$$I_s(h) = 0 \quad . \tag{6.2}$$

In terms of I_s, the back-scattering cross section is

$$\sigma_B = (4\pi)^{-1} \omega^2 \mu_o^2 \left| \int_{-h}^{h} dz \, I_s(z) \right|^2 . \tag{6.3}$$

Within the framework of the approximations used in Part I, this is given by

$$\sigma_B = 4\pi |\Omega_1|^{-2} \, |h + C_s k^{-1} \sin kh|^2 , \tag{6.4}$$

Without the C_s term, this is just the nonresonant formula of Chu [5].

As in the case of the driven antenna, the value of C_s is to be determined from (2.17), with the following form for F,

$$F_s(z) = 4\pi i (\mu_o \omega)^{-1} [1 + C_s \cos k(h - z)] H(2h - z) . \qquad (6.5)$$

It is thus useful to define

$$V'(Z) = \int_{C_o} d\zeta \; L_+(\zeta) \int_0^{\infty} dz \exp(-i\zeta z) [H(Z - z) - \exp(-ikz)] . \qquad (6.6)$$

Thus C_s is given by

$$C_s = - V'(2h)/[\cos kh \, T'(2h) + \sin kh \, S'(2h)] , \qquad (6.7)$$

or approximately with (4.11-12)

$$C_s = - V'(2h) L_+(k) / [\cos kh \, T(2h) + \sin kh \, S(2h)] . \qquad (6.8)$$

From (6.6), V' is found to be

$$V'(Z) = 2\pi L_+(0) + i [L_+(k)]^{-1} \int_{-\infty}^{-1} d\zeta \; \zeta^{-1} \exp(-i\zeta k Z)$$

$$\left\{ [2\Omega_o - \ln(\zeta^2 - 1)]^{-1} - [2\Omega_o - \ln(\zeta^2 - 1) + 2\pi i]^{-1} \right\} . \qquad (6.9)$$

For kZ large, this is approximately

$$V'(Z) L_+(k) = 2\pi L_+(0) L_+(k) + 2 e^{ikZ} \left\{ [\Omega_2(Z)]^{-1} - [\Omega_3(Z)]^{-1} \right\} . \qquad (6.10)$$

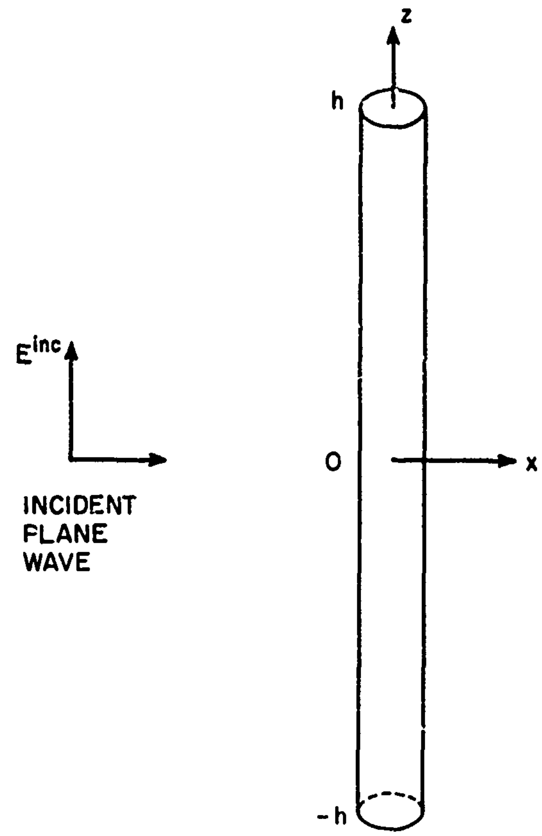

FIG. 3 THE RECEIVING DIPOLE OF HALF-LENGTH h

To avoid numerical integration, it is convenient to use

$$\bar{L}_{+}(0)\,\bar{L}_{+}(k) \sim (2\Omega_1)^{-1}. \tag{6.11}$$

The explicit formula for the back-scattering cross section is given by (6.4), with (6.8) and (6.10-11).

7. The Two-Wire Line

In principle, the method of Part I is applicable to the case of a system of two identical, nonstaggered, parallel dipole antennas as shown in Fig. 4 provided that it is admissible to assume that the current distribution is rotationally symmetrical on each dipole. This assumption is reasonable if the antennas are thin and if the separation b between the antennas is not too small. If the symmetrical and antisymmetrical parts of the curents are used to set up integral equations, then the present case differs from the case of a single dipole only in the appearance of a more complicated kernel. For example, the kernel for the antisymmetrical part of the current is

$$K_a(z) = K(z) - (z^2 + b^2)^{-1/2} \exp[ik(z^2 + b^2)^{1/2}], \tag{7.1}$$

with the Fourier transform

$$\bar{K}_a(\zeta) = \bar{K}(\zeta) - \pi i H_o^{(1)}[b(k^2 - \zeta^2)^{1/2}]. \tag{7.2}$$

The kernel K_s for the symmetric part of the current differs only in a sign.

Besides $\underline{a}$ this problem is characterized by three lengths: λ, h, and b. In the general case, it seems difficult to get easily computable formulas from these kernels. When $kb \ll 1$, the $H_o^{(1)}$ in the Fourier transforms of the

kernels may be replaced by a logarithm, analogous to (3.1). In particular

$$\overline{K}_s(\zeta) = 2\left\{2\Omega_{1s} - \ln\left[(k^2 - \zeta^2)/k^2\right]\right\} , \tag{7.3}$$

where

$$\Omega_{1s} = \ln\left[2/(k^2 a b)^{1/2}\right] - \gamma , \tag{7.4}$$

A well-known result follows immediately from (7.3-4), namely, that the equivalent radius of this antenna system is $(ab)^{1/2}$ [K, p. 275]. In more general cases, the present point of view reproduces all the results of Harrison and King [5] on effective radii.

In the remainder of this section, the following special situation of the antisymmetrical case is to be considered

$$kb^2/\lambda \ll 1 . \tag{7.5}$$

Furthermore, attention is to be restricted entirely to the approximate determination of the total power radiated and the capacitive end correction, both topics outside of the realm of conventional transmission line theory. An important difference between the present case and that of a single dipole antenna is that $\overline{K}_a(\zeta)$ is bounded in the vicinity of $\zeta = -k$ while $\overline{K}(\zeta)$ is not. Consequently,

$$\overline{L}_+(-k) = 0 \qquad \text{but} \qquad \overline{L}_{+a}(-k) \neq 0 . \tag{7.6}$$

Note that $\overline{L}_{\pm a}(\zeta)$ etc. are defined analogous to $\overline{L}_{\pm}(\zeta)$ etc. except that $\overline{K}_a(\zeta)$ is used instead of $\overline{K}(\zeta)$. Because of (7.6), (4.4) cannot be generalized to the present case without modification. Instead, define

$$W_{\pm}'(Z) = \int_{C_o} d\zeta\, \overline{L}_{+a}(\zeta)\left[\int_0^Z dz \exp(-i\zeta z \pm ikz) + \int_{-\infty}^0 dz \exp(-i\zeta z - ikz)\right]. \tag{7.7}$$

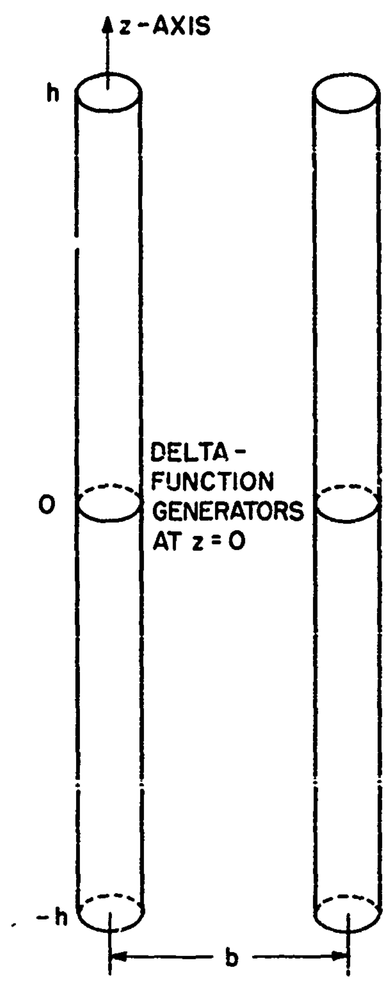

FIG. 4 TWO IDENTICAL, PARALLEL NON-STAGGERED ANTENNAS OF HALF-LENGTH h

If for large Z only terms of the orders Z^{0} and Z^{-1} are kept, W' may be evaluated approximately to be

$$W_{+}'(Z) = 2\pi \overline{L}_{+a}(k) , \tag{7.8}$$

and

$$W_{-}'(Z) = 2\pi \overline{L}_{+a}(-k) \left[1 - \frac{ikb^{2}}{4Z \ln(b/a)}\right]. \tag{7.9}$$

The constant C_a in the integral equation for the current,

$$\int_{-h}^{h} dz' I_a(z') K_a(z-z') = 4\pi i \zeta_0^{-1} \left[C_a \cos kz + \frac{1}{2} \sin k |z|\right], \tag{7.10}$$

is determined from the boundary condition $I_a(h) = 0$. It follows from (4.6) and (7.7) that

$$C_a = \frac{1}{2} i \left[e^{ikh} W_{-}'(2h) + e^{-ikh} W_{+}'(2h)\right]^{-1}$$

$$\left\{e^{ikh} \left[2W_{-}'(h) - W_{-}'(2h)\right] - e^{-ikh}\left[2W_{+}'(h) - W_{+}'(2h)\right]\right\}, \tag{7.11}$$

In this formula, insofar as $\overline{L}_{+a}(\zeta)$ is concerned, only the ratio

$$\overline{L}_{+a}(-k)/\overline{L}_{+a}(k) = \Gamma \exp(2ikh_c) \tag{7.12}$$

enters. Eq. (7.12) defines the real numbers Γ and h_c. The quantity h_c is to be interpreted as the apparent change in line length due to the capacitive end correction. In the language of transmission line theory, h_c is equal to the

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apparent terminal capacitance divided by the capacitance per unit length of the infinite line. The substitution of (7.12) into (7.11) gives

$$C_a = -\frac{1}{2} i \left[1 + \left(1 - \frac{i\alpha}{2kh}\right)\Lambda\right]^{-1}\left[1 - \left(1 - \frac{3i\alpha}{2kh}\right)\Lambda\right], \tag{7.13}$$

where

$$\alpha = \frac{1}{4} k^2 b^2 / \ln(b/a), \tag{7.14}$$

and

$$\Lambda = \Gamma \exp\left[2ik(h + h_c)\right]. \tag{7.15}$$

In terms of C_a, the input admittance is [7]

$$Y_a = 2i\zeta_o^{-1}\left[S_a(h) + C_a U_a(h)\right], \tag{7.16}$$

where S_a and U_a are analogous to (4.1) and (4.10) respectively:

$$S_a(Z) = \int_{C_o} d\zeta \left[\bar{K}_a(\zeta)\right]^{-1} \int_o^Z dz \sin kz \exp(-i\zeta Z), \tag{7.17}$$

and

$$U_a(Z) = \int_{C_o} d\zeta \left[\bar{K}_a(\zeta)\right]^{-1} \int_{-Z}^{Z} dz \cos kz \exp(-i\zeta z). \tag{7.18}$$

It may be noted that (7.16) is correct only to $(kh)^{-1}$. The reason is that a term of the order $(kh)^{-2}$ must be added to correct for the fact that $A(z)$ does not vanish for $|z| > h$. When $kZ \gg 1$, the function U_a of (7.18) may be evaluated by the procedure used to derive (7.8-9):

$$U_a(Z) = \pi\left[\ln(b/a)\right]^{-1}\left[1 - i\alpha/(kZ)\right]. \tag{7.19}$$

The radiation conductance G^e is just $2\,\mathrm{Re}\,Y_a$, since the driving voltage has been taken to be 1. The factor 2 here comes from the fact that there are two wires. It follows from (7.16) that

$$G^e = -4\,\zeta_o^{-1}\,[\mathrm{Im}\,S_a(h) + \mathrm{Im}\,C_a\,U_a(h)]\,. \tag{7.20}$$

This is fortunate since the imaginary part of S_a is simpler than the real part.

So far only the simplifying assumption (7.5) has been used. In order to get explicit and useful answers, the further assumption that $kb \ll 1$ is to be made. In this limit, the first few terms for Γ, h_c and $\mathrm{Im}\,S_a(h)$ are found in Appendix B to be

$$\ln \Gamma = \frac{-k^2 b^2}{4\ln(b/a)}\left\{1 - \frac{k^2 b^2}{6}\left[\frac{1}{4} - \frac{\ln(kb) + \gamma - 11/6}{\ln(b/a)}\right]\right\}, \tag{7.21}$$

$$h_c = -\frac{b}{\pi}\int_0^{\infty}\frac{d\xi}{\xi^2}\,\ln\frac{I_o(a\xi/b)\,K_o(a\xi/b) - K_o(\xi)}{\ln(b/a)}, \tag{7.22}$$

and

$$\mathrm{Im}\,S_a(h) = -\frac{\pi}{2}\,\frac{k^2 b^2}{4[\ln(b/a)]^2}\left\{1 - \frac{k^2 b^2}{6}\left[\frac{1}{4} - 2\,\frac{\ln(kb) + \gamma - 11/6}{\ln(b/a)}\right]\right\}. \tag{7.23}$$

The substitution of (7.13), (7.19) and (7.23) into (7.20) gives

$$G^e = \frac{2\pi}{\zeta_o \ln(b/a)}\left\{-\frac{2}{\pi}\ln\frac{b}{a}\,\mathrm{Im}\,S_a(h) + \frac{1 - \Gamma^2 - 3\alpha\,\Gamma\sin 2k(h+h_c)/kh}{1 + \Gamma^2 + \Gamma\,[2\cos 2k(h+h_c) + \alpha\sin 2k(h+h_c)/kh]}\right\}. \tag{7.24}$$

The term "near resonance" shall be used to refer to the situation where $|C_a|$ in (7.20) is of the order of magnitude $(kb)^{-2}$. Near resonance, the first term in the braces of (7.24) is smaller than the second by a factor of the order $(kb)^{-4}$, and hence may be neglected. As a function of h, the second term shows sharp maxima in the vicinity of $k(h + h_c) \sim (n + \frac{1}{2})\pi$. From the point of view of carrying out experiments on the power radiation from a two-wire line, the widths of these resonances are of interest. Let δ be the total half-power width, i.e., the interval on the h-axis where G^e is larger than half of the maximum value of G^e, then

$$k\delta = -\ln\Gamma, \tag{7.25}$$

independent of n. This is accurate to the order $(kh)^{-1}$ but not $(kh)^{-2}$.

In (7.22), h_c is expressed in terms of an integral with the relative error $(kb)^2$. This integral remains to be evaluated numerically. When $\ln(b/a) \gg 1$, a condition almost never fulfilled in practice, an approximate evaluation is possible:

$$h_c \sim (b/\pi)\,[\ln(b/a)]^{-1} \int_0^\infty d\xi \quad \xi^{-2}\,\{K_0(\xi) - \ln(2/\xi) - \gamma\}. \tag{7.26}$$

This integral can be evaluated by shifting the contour of integration and applying the Weber-Schafheitlin integral:

$$h_c \sim \frac{1}{2}\, b/\ln(b/a) \,. \tag{7.27}$$

This formula is due to King[8], whose derivation is much simpler.

Equations (7.22) and (7.24) give the required answers on the two-wire line. However, it is desirable to calculate the so-called radiation resistance for comparison with the theory of Storer and King[9]. Attention will be restricted to the case $kb \ll 1$ and $kh \gg 1$. The radiation resistance R^e is defined as G^e divided by the square of a "maximum" current. In the theory of Storer and King, a sinusoidal current is assumed on the two-wire line, and thus this definition is meaningful. From the present point of view, the various current "maxima" are of slightly different size and thus R^e is not precisely defined. Let

$$I_{max} = 2\zeta_o^{-1} \left(|C|^2 + \frac{1}{4} \right)^{\frac{1}{2}} \pi / \ln(b/a) \, . \tag{7.28}$$

Up to $(kh)^{-1}$, this approximates the maximum currents near the driving point with an error of the order $(kb)^{-2}$ in general, but of the order of $(kb)^{-4}$ near resonance. With the definition

$$R^e = I_{max}^{-2} \, G^e \, , \tag{7.29}$$

R^e is given by

$$R^e = \pi^{-1} \zeta_o \ln(b/a) \left[1 + \Gamma^2 - \alpha\Gamma \sin 2k(h + h_c) / (kh) \right]^{-1}$$

$$\Big\{ \left[-(2/\pi) \ln(b/a) \, \mathrm{Im}\, S_a(h) \right] \left[1 + \Gamma^2 + 2\Gamma \cos 2k(h + h_c) + \alpha\Gamma \sin 2k(h + h_c)/(kh) \right]$$

$$+ 1 - \Gamma^2 - 3\alpha\Gamma \sin 2k(h + h_c)/(kh) \Big\} \, . \tag{7.30}$$

The leading term of this is

$$R^e = (4\pi)^{-1} \zeta_o k^2 b^2 \left[2 + \cos 2k(h+h_c) - \frac{3 \sin 2k(h+h_c)}{2kh}\right]. \tag{7.31}$$

Except for the end correctin h_c, this is derived by the method of Storer and King in Appendix C. Near resonance, however, the present procedure gives more information. Here the first term in the braces of (7.30) may be neglected. Since $\Gamma^{-1} - \Gamma \sim -2 \ln \Gamma$ and $\Gamma^{-1} + \Gamma \sim 2$, (7.30) reduces near resonance to

$$R^e = \pi^{-1} \zeta_o \ln(b/a) \left[1 + \alpha \frac{\sin 2k(h+h_c)}{2kh}\right] \left[-\ln \Gamma - 3\alpha \frac{\sin 2k(h+h_c)}{2kh}\right]. \tag{7.32}$$

Furthermore, near resonance the quantity $\sin 2k(h+h_c)$ is of the order of $(kb)^2$ so that the first bracket may be neglected. The substitution of (7.14) and (7.21) into (7.32) gives finally

$$R^e = (4\pi)^{-1} \zeta_o (kb)^2 \left\{1 - \frac{k^2 b^2}{6} \left[\frac{1}{4} - \frac{\ln(kb) + \gamma - 11/6}{\ln(b/a)}\right] - \frac{3 \sin 2k(h+h_c)}{2kh}\right\}. \tag{7.33}$$

Right at the point of maximum power radiation, the last term is negligible with the result

$$R^e = -\pi^{-1} \zeta_o \ln(b/a) \ln \Gamma$$

$$= (4\pi)^{-1} \zeta_o (kb)^2 \left\{1 - \frac{k^2 b^2}{6} \left[\frac{1}{4} - \frac{\ln(kb) + \gamma - 11/6}{\ln(b/a)}\right]\right\}. \tag{7.34}$$

Note that, from (7.25), (7.34) may be written as

$$R^e = k\delta R_c \tag{7.35}$$

where R_c is the characteristic resistance of the infinite two-wire line. Eq. (7.34) has been previously reported[10].

It should be emphasized that all results in this section depend on the initial assumption of a rotationally symmetrical current distribution on each dipole. In the language of transmission-line theory, the proximity effect of the two wires has been neglected. This leads to a relative error of the order of $(a/b)^2$ in all the physical quantities studied.

8. The Dielectric-Coated Antenna

A dipole antenna with a thin layer of dielectric material on the outside has many interesting properties. When the antenna is relatively short, its behavior does not differ much from the dipole without the dielectric. When it is relatively long, it behaves more like a transmission line than an antenna. In this section, this dielectric-coated antenna is to be studied only from the point of view of illuminating certain essential points of the procedure used in this paper. Although this problem is an extremely interesting one, no quantitative calculations will be made since no systematic experimental data seem to be available for this type of antenna.

The present procedure depends on the solution of a Wiener-Hopf integral equation. Therefore, it is essential that the geometry of the problem be translationally invariant in the direction of the antenna after the removal of the perfectly conducting dipole antenna. Accordingly, in studying the dielectric-coated antenna, the dielectric layer is assumed to extend to infinity in the

$\pm$ z directions. When the dielectric tube is sufficiently small in cross section to support no "mode," these extensions may be expected to make no significant difference. The geometry of the assumed model is shown in Fig. 5. To avoid unnecessary complications, the dielectric material is assumed to be nonmagnetic with a dielectric constant $\epsilon = \epsilon_o \epsilon_r > \epsilon_o$.

The problem of setting up an integral equation analagous to (1.2) is not entirely straightforward. It follows from the time-independent Maxwell's equations that for this geometry a current distribution

$$\underset{\sim}{J} = \hat{z}\,\delta(r-a)\exp(-i\zeta z) \tag{8.1}$$

leads to the following electric field at $r = a$:

$$E_z(a,z) = a(i\omega\epsilon_o)^{-1} G(\zeta)\exp(-i\zeta z)\,, \tag{8.2}$$

where

$$G(\zeta) = (k^2-\zeta^2)\left\{\ln\left[(k^2-\zeta^2)^{\frac{1}{2}} b/2\right] + \gamma - \pi i/2\right\} - \epsilon_r^{-1}(k'^2-\zeta^2)\ln(b/a). \tag{8.3}$$

This is approximately valid, in the same sense as (3.1), when

$$k' b << 1, \tag{8.4}$$

where

$$k' = \epsilon_r^{\frac{1}{2}} k\,. \tag{8.5}$$

So far as branch cuts are concerned, $G(\zeta)$ has the same structure as $\bar{K}(\zeta)$. On the real axis, when $\zeta > k$, (8.3) gives

$$G(\zeta) = -(\zeta^2-k^2)\left\{\ln\left[(\zeta^2-k^2)^{\frac{1}{2}} b/2\right] + \gamma\right\} - \epsilon_r^{-1}(k'^2-\zeta^2)\ln(b/a), \tag{8.6}$$

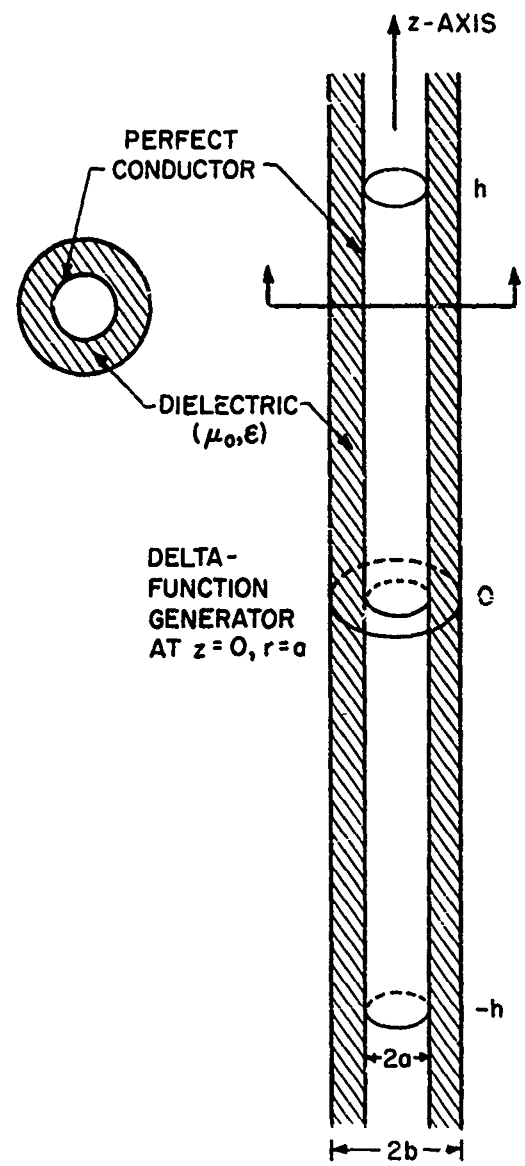

FIG. 5 THE DIELECTRIC COATED ANTENNA OF HALF-LENGTH h

It follows that on the real axis when $\zeta > k$, $G(\zeta)$ is real with one zero between k and k'. Call this zero $\zeta = k_d$. A more accurate calculation indicates that this zero is actually located slightly above the real axis. Thus, within the present approximation, all integrations along the real axis in the ζ-plane should be carried out with the contour C_1 shown in Fig. 6.

Formally, this zero of $G(\zeta)$ at $\zeta = k_d$ leads to

$$\lim_{z \to \infty} \left| \int_{C_1} d\zeta \, \exp(i\zeta z) [G(\zeta)]^{-1} \right| \neq 0. \tag{8.7}$$

However, for the procedure of Part I to work, the inverse Fourier transform of $\overline{M}$ must approach zero sufficiently rapidly as $z \to \infty$. In order to get the kernel $\overline{K}$, it is thus necessary to remove from $G(\zeta)$ the zero at $\zeta = k_d$. The simplest way to remove this zero is to define

$$\overline{K}_d(\zeta) = 2(\zeta^2 - k_d^2)^{-1} G(\zeta). \tag{8.8}$$

With this definition, it follows from (8.2) that in the present case the integral equation for the current is

$$A_d(z) = (4\pi)^{-1} \mu_o \int_{-h}^{h} dz' \, I_d(z') K_d(z - z'), \tag{8.9}$$

where K_d is the inverse Fourier transform of $\overline{K}_d$, and A_d satisfies

$$\left(\frac{d^2}{dz^2} + k_d^2\right) A_d(z) = i\omega\mu_o\epsilon_o \delta(z), \tag{8.10}$$

for $|z| < h$. By symmetry and for $|z| < h$, $A_d(z)$ is given by,

$$A_d(z) = \mu_o i \; \zeta_d^{\;-1} \; (C_d \cos k_d z + \tfrac{1}{2} \sin k_d \, |z|), \tag{8.11}$$

where

$$\zeta_d = (\omega\epsilon_o)^{-1} k_d \, . \tag{8.12}$$

Eqs. (8.9) and (8.11) are analogous to (1.2) and (1.4) respectively. But it should be emphasized that A_d is not the vector potential for this problem. From this point of view, the use of the vector potential in Part I may be considered to be coincidental.

As before, the constant C_d in (8.11) is to be determined from the boundary condition

$$I_d\,(h) = 0 \, . \tag{8.13}$$

Thus, the present problem is formulated in terms of an integral equation entirely analogous to that of the dipole antenna without dielectric coating, the only difference being in the kernel. In principle, the procedure of Part I may be applied here, but the details are somewhat more complicated. Without going into the details, however, several qualitative statements may be made about this antenna by considering the kernel. First, unless b/a is very large, which is not feasible practically, k_d is quite close to k. Thus, the thin dielectric coating makes only a slight modification on the behavior of the dipole antenna unless the antenna is at least several wavelengths long. On the other hand, when z is large, the behavior of $K_d(z)$ is determined almost entirely by that of $\overline{K}_d(\zeta)$ in the vicinity of the singularities at $\zeta = \pm k$. In this vicinity, $\overline{K}_d(\zeta)$ is qualitatively very similar to $\overline{K}_a(\zeta)$ of the last section. Accordingly, a long dielectric-coated antenna behaves like a transmission line.

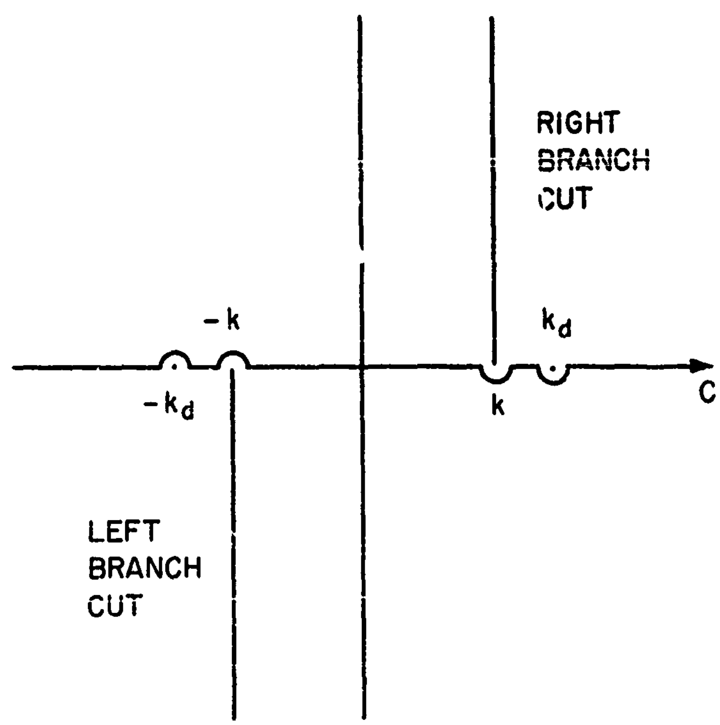

FIG. 6 THE CONTOUR C_1

In particular, $\lim_{h \to \infty} C_d$ in (8.11) does not exist. That is, the end of the antenna has a profound effect on the current distribution near the driving point no matter how long the antenna is. This is typical of a transmission line. This is also in agreement with the experimental observation that any abrupt bend in the antenna causes significant radiation. Moreover, the present point of view makes it possible to make a semi-quantitative statement: a bend causes appreciable radiation unless the radius of curvature of the bend is much larger than $\lambda (k_d / k - 1)^{-1}$.

In order to make a quantitative comparison with the approach making use of a surface impedance [K, p. 28], it remains to define the equivalent radius and the equivalent surface reactance. This is done by writing the $G(\zeta)$ of (8.3) in the form

$$G(\zeta) = (k^2 - \zeta^2) \left\{ \ln [(k^2 - \zeta^2)^{\frac{1}{2}} b/2] - \epsilon_r^{-1} \ln (b/a) + \gamma - \pi i/2 \right\}$$
$$- \epsilon_r^{-1} (k'^2 - k^2) \ln (b/a). \tag{8.14}$$

The second term of (8.14) gives immediately the surface impedance per unit length

$$z^i = -i \omega L, \tag{8.15}$$

where the inductance per unit length L is

$$L = \mu_o (2\pi)^{-1} \ln (b/a) (1 - \epsilon_r^{-1}). \tag{8.16}$$

On the other hand, the first term on the right hand of (8.14) gives the equivalent radius a_d in the form

$$a_d = b (a/b)^{1/\epsilon_r} \tag{8.17}$$

A dipole antenna of radius $\underline{a}$ with a dielectric coating of relative dielectric constant ϵ_r and thickness $\underline{b} - \underline{a}$ is equivalent to a dipole antenna of radius a_d and surface impedance per unit length $-i\omega L$.

Acknowledgment

I am greatly indebted to Professor Ronold W. P. King for introducing me to this subject and for his patient guidance and numerous discussions throughout many years.

Appendix A

This appendix is devoted to the detailed derivation of (4.13-15). It follows from (4.1) that $S(Z)$ has the integral representation

$$S(Z) = -\frac{1}{2}\int_0^{\infty}\Big\{[\,d\xi \;\; [2(\Omega_o - \ln 2) - \ln \xi(\xi+1)]^{-1} - [2(\Omega_o - \ln 2) + 2\pi i - \ln \xi(\xi+1)]^{-1}\Big\}$$
$$\Big\{(1+\xi)^{-1}\exp[2ikZ(1+\xi)] - \xi^{-1}\exp(2ikZ\xi) + [\xi(1+\xi)]^{-1}\Big\}. \tag{A.1}$$

Let Ξ be a large number, then

$$\int_0^{\Xi} d\xi \Big\{[2(\Omega_o - \ln 2) - \ln \xi(\xi+1)]^{-1} - [2(\Omega_o - \ln 2) + 2\pi i - \ln \xi(\xi+1)]^{-1}\Big\}[\xi^{-1} + (\xi+1)^{-1}]$$
$$= \ln\Big\{1 + \pi i[\Omega_o - \ln 2 - \ln \Xi]^{-1}\Big\}, \tag{A, 2}$$

and

$$2\int_0^{\infty} d\xi\Big\{[2(\Omega_o - \ln 2) - 2\ln(\xi+1)]^{-1} - [2(\Omega_o - \ln 2) + 2\pi i - 2\ln(\xi+1)]^{-1}\Big\}(\xi+1)^{-1}$$
$$= \ln\Big\{1 + \pi i[\Omega_o - \ln 2 - \ln \Xi]^{-1}\Big\} - \ln\Big\{1 + \pi i[\Omega_o - \ln 2]^{-1}\Big\}. \tag{A.3}$$

On the other hand, approximately

$$\int_0^\infty d\xi \left\{ [2(\Omega_o - \ln 2) - \ln \xi(\xi+1)]^{-1} - [2(\Omega_o - \ln 2) - 2\ln(\xi+1)]^{-1} \right\} (\xi+1)^{-1}$$

$$\sim - \int_0^\infty d\xi \, [2(\Omega_o - \ln 2) - 2\ln(\xi+1)]^{-2} \ln[(\xi+1)/\xi] (\xi+1)^{-1}$$

$$\sim - \int_0^\infty d\xi \, [2(\Omega_o - \ln 2) - 2\ln 2]^{-2} \ln[(\xi+1)/\xi] (\xi+1)^{-1}$$

$$= -\frac{\pi^2}{6} [2(\Omega_o - 2\ln 2)]^{-2} . \qquad \text{(A. 4)}$$

Similarly,

$$\int_0^\infty d\xi \left\{ [2(\Omega_o - \ln 2) + 2\pi i - \ln \xi(\xi+1)]^{-1} \right.$$

$$\left. - [2(\Omega_o - \ln 2) + 2\pi i - 2\ln(1+\xi)]^{-1} \right\} (\xi+1)^{-1}$$

$$\sim -\frac{\pi^2}{6} [2(\Omega_o - 2\ln 2) + 2\pi i]^{-2} . \qquad \text{(A. 5)}$$

The combination of (A. 2-5) yields

$$\int_0^\infty d\xi \left\{ [2(\Omega_o - \ln 2) - \ln \xi(\xi+1)]^{-1} - [2(\Omega_o - \ln 2) + 2\pi i - \ln \xi(\xi+1)]^{-1} \right\} [\xi(1+\xi)]^{-1}$$

$$-\ln\left\{1 + \pi i [\Omega_o - \ln 2]^{-1}\right\} + \frac{\pi^2}{3} \left\{ [2(\Omega_o - 2\ln 2)]^{-2} - [2(\Omega_o - 2\ln 2) + 2\pi i]^{-2} \right\} . \qquad \text{(A. 6)}$$

It remains to study the other two terms of (A. 1). It follows from (4. 16-18) that

$$\int_0^\infty d\xi \left\{[2(\Omega_o - \ln 2) - \ln\xi(\xi+1)]^{-1} - [2(\Omega_o - \ln 2) + 2\pi i - \ln\xi(\xi+1)]^{-1}\right\}\xi^{-1}\exp(2ikZ\xi)$$

$$\sim \int_0^\infty d\xi\ \xi^{-1}\ e^{-\xi}\left\{[\Omega_2(Z) - (\ln\xi+\gamma)]^{-1} - [\Omega_3(Z) - (\ln\xi+\gamma)]^{-1}\right\}$$

$$= \int_0^\infty d\xi\ e^{-\xi}\ln\left\{[\Omega_2(Z) - (\ln\xi+\gamma)]^{-1}[\Omega_3(Z) - (\ln\xi+\gamma)]\right\}$$

$$\sim \ln\left\{[\Omega_2(Z)]^{-1}\Omega_3(Z)\right\} + \frac{1}{2}\gamma'\left\{[\Omega_2(Z)]^{-2} - [\Omega_3(Z)]^{-2}\right\} . \qquad (A.7)$$

Also,

$$\int_0^\infty d\xi \left\{[2(\Omega_o - \ln 2) - \ln\xi(\xi+1)]^{-1}\right.$$

$$\left. - [2(\Omega_o - \ln 2) + 2\pi i - \ln\xi(\xi+1)]^{-1}\right\}(1+\xi)^{-1}\exp[2ikZ(1+\xi)]$$

$$\sim i(2kZ)^{-1}\exp(2ikZ)\left\{[\Omega_2(Z)^{-1} - [\Omega_3(Z)]^{-1}\right\} . \qquad (A.8)$$

The substitution of (A.6-8) into (A.1) yields (4.13).

Next, it follows from (4.1-2) that

$$T(Z) - iS(Z) = i\int_{C_o} d\zeta\ \bar{M}(\zeta)\ (\zeta+k)^{-1}\exp[-i(\zeta+k)Z] , \qquad (A.9)$$

or

$$T(Z) - i\,S(Z) = -i \int_0^\infty d\xi \;\; \xi^{-1} \exp(2\,ikZ\,\xi)$$

$$\left\{ [2(\Omega_o - \ell n\, 2) - \ell n\, \xi(1+\xi)]^{-1} - [2(\Omega_o - \ell n\, 2) + 2\pi i - \ell n\, \xi(1+\xi)]^{-1} \right\}. \qquad \text{(A. 10)}$$

Eq. (4. 14) follows from (A. 7) and (A. 10).

Finally, it follows from (4. 10) that

$$U(Z) = -i \int_0^\infty d\xi \left\{ [2(\Omega_o - \ell n\, 2) - \ell n\, \xi(1+\xi)]^{-1} - [2(\Omega_o - \ell n\, 2) + 2\pi i - \ell n\, \xi(1+\xi)]^{-1} \right\}$$

$$\left\{ (1+\xi)^{-1} \exp[i\,2kZ(1+\xi)] + \xi^{-1} \exp(i\,2kZ\,\xi) \right\}. \qquad \text{(A. 11)}$$

Eq. (4. 15) then follows from (A. 7), (A. 8) and (A. 11).

Appendix B

In this appendix (7.21-23) are to be derived. For this purpose, it is convenient to define, with (7.12),

$$\Delta = -\ln \Gamma - 2\,ikh_c = \ln [\overline{L}_{+a}(k) / \overline{L}_{+a}(-k)] \tag{B.1}$$

Since

$$\overline{K}_a(k) = \overline{K}_a(-k) = 2 \ln (b/a), \tag{B.2}$$

it is also convenient to introduce

$$\overline{K}_a{}'(\zeta) = \overline{K}_a(\zeta) / [2 \ln (b/a)]. \tag{B.3}$$

Then,

$$\ln \overline{L}_{+a}(\zeta) = -[\ln \overline{K}_a{}'(\zeta)]_+ - \frac{1}{2} \ln [2 \ln (b/a)]. \tag{B.4}$$

It follows from (B.1) and (B.4) that

$$\Delta = -(\pi i)^{-1} k \int_{C_o} d\zeta\, (\zeta^2 - k^2)^{-1} \ln \overline{K}_a{}'(\zeta). \tag{B.5}$$

But the integrand here is bounded in the neighborhoods of $\zeta = \pm k$. Thus, the contour C_o may be replaced by the real axis:

$$\Delta = \pi^{-1}\, 2\,ik \int_0^\infty d\zeta\, (\zeta^2 - k^2)^{-1} \ln \overline{K}_a{}'(\zeta). \tag{B.6}$$

Since the intergrand is real for $\zeta > k$, only the part $0 < \zeta < k$ contributes to Γ. Write

$$\Delta = \Delta_1 + \Delta_2 , \tag{B.7}$$

where

$$\Delta_1 = \frac{2ik}{\pi} \int_0^k d\zeta \, (\zeta^2 - k^2)^{-1} \ln \frac{\pi i \left\{ J_0 [a(k^2-\zeta^2)^{\frac{1}{2}}] \, H_0^{(1)} [a(k^2-\zeta^2)^{\frac{1}{2}}] - H_0^{(1)} [b(k^2-\zeta^2)^{\frac{1}{2}}] \right\}}{2 \ln (b/a)} , \tag{B.8}$$

and

$$\Delta_2 = \frac{2ik}{\pi} \int_k^\infty d\zeta \, (\zeta^2 - k^2)^{-1} \ln \frac{I_0[a(\zeta^2-k^2)^{\frac{1}{2}}] \, K_0[a(\zeta^2-k^2)^{\frac{1}{2}}] - K_0[b(\zeta^2-k^2)^{\frac{1}{2}}]}{\ln (b/a)} . \tag{B.9}$$

First Δ_2 is to be calculated to the accuracy $(kb)^2$. The change of variable

$$\xi = b(\zeta^2 - k^2)^{\frac{1}{2}} \tag{B.10}$$

gives

$$\Delta_2 = \pi^{-1} \, 2ikb \, (S_1 + S_2), \tag{B.11}$$

where

$$S_1 = \int_0^\infty d\xi \; \xi^{-2} \ln \left\{ [\ln b/a]^{-1} [I_0(a\xi/b) \, K_0(a\xi/b) - K_0(\xi)] \right\} \tag{B.12}$$

depends only on the ratio b/a, and

$$S_2 = \int_0^\infty d\xi \; \xi^{-1} [(\xi^2 + k^2b^2)^{-\frac{1}{2}} - \xi^{-1}] \ln \left\{ [\ln b/a]^{-1} [I_0(a\xi/b) \, K_0(a\xi/b) - K_0(\xi)] \right\} . \tag{B.13}$$

S_2 may be determined approximately by expanding the logarithmic factor in powers of ξ. If $\xi' = \xi(kb)^{-1}$, then

$$S_2 = kb\,[4\,\ell n(b/a)]^{-1} \int_0^\infty d\xi'\ \xi'\,[(\xi'^2+1)^{-\frac{1}{2}} - \xi'^{-1}]\,[\gamma - 1 + \ell n\,(kb\xi'/2)]\,. \tag{B.14}$$

This gives finally

$$S_2 = kb\,[4\,\ell n\,(b/a)]^{-1}\,[2 - \gamma - \ell n\,(kb)]\,. \tag{B.15}$$

In order to get any new results on the radiation resistance of the two-wire line, it is necessary to calculate Γ to the accuracy $(kb)^4$. Therefore, the real part of Δ_1 should be calculated to the accuracy $(kb)^4$, the imaginary part only to $(kb)^2$. For this purpose, write Δ_1 in terms of the new variable of integration $\xi = (k^2 - \zeta^2)^{1/2}$:

$$\Delta_1 = -\pi^{-1} 2ik \int_0^k d\xi\ \xi^{-1}\,(k^2 - \xi^2)^{-\frac{1}{2}}\,\ell n\left\{\pi i\,[2\,\ell n(b/a)]^{-1}[J_o(a\xi)H_o^{(1)}(a\xi) - H_o^{(1)}(b\xi)]\right\}. \tag{B.16}$$

Now the logarithm may be expanded, keeping terms up to b^4, but neglecting a^2. This gives the result

$$\Delta_1 = \frac{-2ik}{\pi} \int_0^k \frac{d\xi}{\xi}\,(k^2 - \xi^2)^{-\frac{1}{2}}$$

$$\left\{(\ell n\,\frac{b}{a})^{-1}\left[-\,(\gamma - 1 + \ell n\,\frac{b\xi}{2} - \frac{\pi i}{2})\,(\frac{b^2\,\xi^2}{4} - \frac{b^4\,\xi^4}{64}) - \frac{b^4\,\xi^4}{128}\right]\right.$$

$$\left. -\,\frac{1}{2}\,(\ell n\,\frac{b}{a})^{-2}\,(\gamma - 1 + \ell n\,\frac{b\xi}{2} - \frac{\pi i}{2})^2\,\frac{b^4\,\xi^4}{16}\right\}. \tag{B.17}$$

The rest of the calculation is tedious but straightforward. Because of the difference of accuracy required in the real and imaginary parts, it is advantageous to write them separately. The real part gives (7.21) and the imaginary part turns out to be simply

$$\operatorname{Im} \Delta_1 = -\pi^{-1}\, 2kb\, S_2 \,, \tag{B.18}$$

Eqs. (B.11) and (B.18) may be combined to yield

$$\operatorname{Im} \Delta = 2\, kb\, S_1 / \pi \,, \tag{B.19}$$

which is the same as (7.22).

In order to get (7.23), write the $S_a(Z)$ of (7.17) in the form

$$S_a(Z) = S_{a1}(Z) + S_{a2}(Z) \,, \tag{B.20}$$

where

$$S_{a1}(Z) = \frac{1}{2} \int_{C_o} d\zeta \left\{ [\overline{K}_a(\zeta)]^{-1} - [\overline{K}_a(k)]^{-1} \right\} [(k-\zeta)^{-1} + (k+\zeta)^{-1}] \,. \tag{B.21}$$

and

$$S_{a2}(Z) = -\frac{1}{2} \int_{C_o} d\zeta \left\{ [\overline{K}_a(\zeta)]^{-1} - [\overline{K}_a(k)]^{-1} \right\} \left\{ (k-\zeta)^{-1} \exp[i(k-\zeta)Z] + (k+\zeta)^{-1} \exp[-i(k+\zeta)Z] \right\} . \tag{B.22}$$

To the order $(kZ)^{-1}$, $S_{a2}(Z)$ is real. Therefore, approximately

$$\operatorname{Im} S_a(Z) = \operatorname{Im} S_{a1}(Z) \,. \tag{B.23}$$

Similar to the case of $\ln \Gamma$, the imaginary part of $S_{a1}(Z)$ comes only from the range $0 < \xi < k$. By series expansion, it follows from (B. 21) and (B. 23) that

$$\operatorname{Im} S_a(Z) = -k\,[\ln(b/a)]^{-1} \operatorname{Im} \int_0^k d\xi \quad \xi^{-1} (k^2 - \xi^2)^{-\frac{1}{2}}$$

$$\left\{ (\ln \frac{b}{a})^{-1} \left[-(\gamma - 1 + \ln \frac{b\xi}{2} - \frac{\pi i}{2}) (\frac{b^2 \xi^2}{4} - \frac{b^4 \xi^4}{64}) - \frac{b^4 \xi^4}{128} \right] - (\ln \frac{b}{a})^{-2} (\gamma - 1 + \ln \frac{b\xi}{2} - \frac{\pi i}{2})^2 \frac{b^4 \xi^4}{16} \right\}. \tag{B. 24}$$

This integrand differs from that of (B. 17) only in the absence of 1/2 . Thus, (7. 23) follows.

Appendix C

In order to derive (7.31) by the method used by Storer and King, the current distribution is assumed to be

$$I(z) = \sin k(h - |z|). \tag{C.1}$$

The tangential component of the vector potential at large distance is immediately verified to be

$$A_{tan}(\theta, \phi) = \text{Constant} [\cos(kh \cos \theta) - \cos kh] \cos \phi , \tag{C.2}$$

where the constant is independent of k, h, θ, and ϕ. Integration over θ and ϕ leads to

$$R^e = \text{Constant} [1 + 2 \cos^2 kh - \frac{3 \sin 2kh}{2kh}]. \tag{C.3}$$

The constant here may be determined by the observation that at resonance [i.e., $kh = (n + \frac{1}{2})\pi$] the present case is identical to that treated by Storer and King. Except for h_c, this gives (7.31).

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